Written Exam at the Department of Economics

Summer 2019

Derivatives Pricing

Final Exam

June 14, 2019

3 hours, open book

Answers in English only

This exam consists of 5 pages in total

Falling ill during the exam

If you fall ill during an examination at Peter Bangs Vej, you must:

- Contact an invigilator who will show you how to register and submit a blank exam paper.
- Leave the examination.
- Contact your GP and submit a medical report to the Faculty of Social Sciences no later than five (5) days from the date of the exam.

Be careful not to cheat at the exam

You cheat at an exam, if during the exam, you:

- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Guidelines:

- The exam is composed of 4 problems, each carrying an indicative weight.
- If you lack information to answer a question, please make the necessary assumptions.
- Please clearly state any assumptions you make.
- All answers must be justified.

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You've joined the options trading desk in the leading Nordic investment bank D.B. as a junior analyst and are just starting your first day on the new job. The managing director of the team is very delighted to have you aboard and already needs your help to solve a number of option pricing problems. Eager to prove yourself to your new colleagues, you immediately start solving the problems.

Problem 1 (20%)

One of the young traders in your new team tells you that a 1-year at-the-money vanilla put option on a non-dividend paying stock S is bid at a price of 10 in the market. The trader's gut feeling tells her it is too expensive, and she decides to sell the option at the bid. She anticipates that the realized volatility of the stock will be 10% and delta-hedges the option continuously at this level of volatility using the Black-Scholes (BSM) model. Assume interest rates are zero, and the spot price of the underlying stock is $S_0 = 100$.

a) Show that the total implied volatility of an at-the-money vanilla option is approximately

$$\Sigma\sqrt{\tau} \approx \sqrt{2\pi}\frac{P}{S_0}$$

where Σ is the implied volatility, P is the option price, and τ is the time to expiration. (*Hint: In the BSM option price formula, Taylor expand the standard normal distribution function to first order:* $N(x) \approx N(0) + N'(0)(x - 0)$.)

b) If the realized volatility turns out to be 10% over the year, what is the trader's final PnL on the short option trade?

After the year had passed, it turned out the trader was wrong in her guess on the realized volatility of the stock. In fact, realized volatility was a whopping 30% and not the trader's anticipated 10%. Moreover, the terminal stock price was $S_T = 95$ at expiration of the option.

c) What is the trader's final PnL of the short option trade?

d) If the trader instead had delta-hedged the option at its implied volatility, what is then the final PnL?

e) If the trader did not delta-hedge at all, what is then the final PnL?

Problem 2 (30%)

The bank has just implemented the Heston stochastic volatility model that it wants to use for option pricing. Unfortunately, the other traders in the team are not very familiar with this model and prefer the classic Black-Scholes model. Therefore, you are asked to provide an analysis of the Heston model in order to make them comfortable with it.

For zero interest rates and dividends, the risk-neutral dynamics of the Heston model for a stock index S_t can be written as

$$dS_t = \sqrt{v_t} S_t dW_t$$
$$dv_t = \lambda(\theta - v_t)\mathrm{d}t + \epsilon\sqrt{v_t} dZ_t$$
$$dW_t dZ_t = \rho \mathrm{d}t,$$

where $\lambda, \theta, \epsilon$ are positive parameters, $\rho \in [-1; 1]$, and $v_0 > 0$ is the initial instantaneous variance. After calibration to market prices of vanilla options, you get the following parameter values for the model $\lambda = 1.15, \theta = 0.02, \epsilon = 0.2,\ \rho = -0.4$ and $v_0 = 0.04$. The spot stock price is $S_0 = 120$.

- **a)** Is the calibrated model consistent with the so-called 'leverage effect'?
- **b)** Derive the minimum-variance delta of a long vanilla put option in this model.
- **c)** If you continuously delta-hedge a long 1-month out-of-the-money put option using the BSM delta, do you then under-hedge, over-hedge or correctly hedge your position?

Based on past empirical behavior of implied volatilities, your boss thinks the speed of mean reversion looks too high from the calibration.

- **d)** What is the impact of mean reversion in variance on the implied volatility surface generated by the model?
- **e)** Using vanilla options, devise a trading strategy that profits if the speed of mean reversion decreases significantly, everything else equal.

If the volatility of volatility is $\epsilon = 0$ and the other parameters are unchanged, the model reduces to a local volatility model with risk-neutral dynamics

$$dS_t = \sigma(t) S_t dW_t$$

- **f)** Determine the local volatility function $\sigma(t)$ and calculate the local volatility in 1 year.
- **g)** Derive an explicit expression for implied volatility in this local volatility model.

You have obtained the following mid prices of vanilla call options from your broker:

Strike	Expiry (years)	Call price
115	1.00	11.095
120	1.00	8.537
120	1.05	8.710
125	1.00	6.441

h) By pricing a calendar and butterfly spread, estimate the at-the-money local volatility in 1 year using Dupire's equation and assess if this value is consistent with your local volatility model.

Problem 3 (30%)

Instead of the Heston model, you boss recommends a simpler stochastic volatility model. For zero dividends and the risk-free interest rate equal to zero, the model has risk-neutral dynamics

$$\begin{aligned} dS_t &= \sqrt{v_t} S_t dW_t \\ dv_t &= \gamma v_t \mathrm{dt} + \epsilon v_t dZ_t \\ dW_t dZ_t &= \rho \mathrm{dt}, \end{aligned}$$

where γ, ϵ are positive parameters, $\rho = 0$, and $v_0 > 0$ is the initial instantaneous variance.

a) Explain how the risk-neutral measure is obtained when volatility is stochastic.

b) Derive the partial differential equation (PDE) and boundary condition satisfied by the arbitrage-free price of a European option $C(t, S, v)$ on S in this model.

c) Find the risk-neutral distribution of the instantaneous volatility $\sqrt{v_t}$.

d) If $\gamma = 0, \epsilon = 0.2$ and $v_0 = 0.04$, calculate the probability that the instantaneous volatility is larger than 20% in 1 year.

A competing bank is offering variance swaps to their clients. To not lose edge in the Scandinavian market, your boss asks you to investigate how to price these derivatives in the stochastic volatility model above. Recall, the payoff of the variance swap with continuous sampling is given by

$$\left(\frac{1}{T}\int_0^T v_t dt - V_K\right)$$

where V_K is the fair value of the variance swap and T is the time to expiry.

e) Under the risk-neutral measure, show that the fair value is equal to the price of a short log contract:

$$V_K = \mathrm{E}^{\mathrm{Q}}\left[\frac{1}{T}\int_0^T v_t dt\right] = -\frac{2}{T}\mathrm{E}^{\mathrm{Q}}\left[\ln\left(\frac{S_T}{S_0}\right)\right]$$

f) Show how to replicate the fair value V_K of the variance swap by a portfolio of vanilla options.

g) Find an explicit expression for the fair value V_K in this stochastic volatility model.

Finally, to check your pricing of the variance swap, you calibrate both the stochastic volatility model and a local volatility model to vanilla option prices. As it turns out, both models fit perfectly the implied volatility smile at time T expiry.

h) Discuss in which of the two models the fair value of the variance swap is largest.

Problem 4 (20%)

After investigating stochastic volatility models, your boss asks you to analyze the impact of jumps in the stock price. Assuming zero interest rates and dividends, recall that in the BSM model the risk-neutral dynamics of the log stock price $X_t = \ln(S_t)$ is given by

$$dX_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t$$

with $\sigma > 0$. Extending the model to allow for jumps in the stock price, the risk-neutral dynamics becomes

$$dX_t = \alpha \mathrm{d}t + \sigma dW_t + J\mathrm{d}q_t$$

where dq_t is a Poisson process and J is a fixed constant jump size.

- **a)** Why is the risk-neutral drift α of the jump-diffusion model different from the BSM drift?
- **b)** Discuss whether jumps have the largest impact for short-expiry or long-expiry vanilla options.
- **c)** Compared to a stochastic volatility model, why might a jump-diffusion model better capture the implied volatility skew of short-expiry equity options?

You decide to calibrate the model to vanilla option prices and find that $\sigma = 25\%$, the jump size is -0.13, and a jump in the stock price occurs once every two months, on average. The current spot price is $S_0 = 100$.

- **d)** What is the probability of observing at least 1 jump in a month?
- **e)** Calculate the price and implied volatility of a 1-month at-the-money call option, truncating the sum in the pricing formula to 2 jumps

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